The defining character of a spring is that it resists displacement from its rest position with a force which increases linearly, by Hooke’s law, the restoring force in the spring, \( F_k \), is proportional to the displacement of the mass:

\[
F_k = -ky
\]

since the mass is travelling through friction at relatively slow speeds, then the equation for the force of linear drag is appropriate and the force is proportional to the velocity of the mass:

\[
F_b = -by'
\]

By Newton’s second law, \( F = ma \) where \( a = y'' \)

\[
F = F_b + F_k \quad \therefore \quad my'' = -by' - ky
\]

\[
my'' + by' + ky = 0
\]

is a linear ODE with constant coefficients.

then the auxiliary or characteristic equation:

\[
u^2 + bu + k = 0
\]

with roots \( u_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \)
Cases:

1) \( b^2 - 4mk > 0 \) (overdamping) roots \( u_{1,2} \) are real, and the general solution
\[
y = c_1 e^{u_1 x} + c_2 e^{u_2 x}
\]
From IVP: \( y(0) = y_0 = 0 \) and \( y'(0) = y'_0 = v_0 \)
\[
y = c_1 e^{u_1 x} + c_2 e^{u_2 x} \quad \therefore \quad 0 = c_1 e^0 + c_2 e^0
\]
\[
y' = c_1 u_1 e^{u_1 x} + c_2 u_2 e^{u_2 x} \quad \therefore \quad v_0 = c_1 u_1 e^0 + c_2 u_2 e^0 \quad \text{and} \quad c_1 = -c_2
\]
\[
\therefore \quad c_1 = \frac{v_0}{u_1 - u_2} \quad \text{and} \quad c_2 = \frac{v_0}{u_2 - u_1}
\]
\[
y = \frac{v_0}{u_1 - u_2} e^{u_1 x} + \frac{v_0}{u_2 - u_1} e^{u_2 x}
\]
In this case the real roots \( u_{1,2} \) are always negative because \( b > \sqrt{b^2 - 4mk} \) in the quadratic formula. Therefore the terms \( e^{u_1 x} \) and \( e^{u_2 x} \) become transient terms as \( x \) goes to infinity.

2) \( b^2 - 4mk = 0 \) (critical damping) roots \( u_1 = u_2 = -\frac{b}{2m} \) and the general solution
\[
y = c_1 e^{\frac{-b}{2m} x} + c_2 xe^{\frac{-b}{2m} x}
\]
From IVP: \( y(0) = y_0 = 0 \) and \( y'(0) = y'_0 = v_0 \)
\[
y = c_1 e^{\frac{-b}{2m} x} + c_2 xe^{\frac{-b}{2m} x} \quad \therefore \quad 0 = c_1
\]
\[
y' = c_1 \frac{-b}{2m} e^{\frac{-b}{2m} x} + c_2 \frac{-b}{2m} xe^{\frac{-b}{2m} x} + c_2 e^{\frac{-b}{2m} x} \quad \therefore \quad v_0 = c_2 \quad \text{and} \quad c_1 = 0
\]
\[
y = v_0 xe^{\frac{-b}{2m} x}
\]

3) \( b^2 - 4mk < 0 \) (underdamping) roots \( u_{1,2} = -\frac{b}{2m} \pm i \frac{\sqrt{b^2 - 4mk}}{2m} = -\alpha \pm \omega \)
\[
y = e^{-\alpha x}(c_1 \cos \omega x + c_2 \sin \omega x)
\]
From IVP: \( y(0) = y_0 = 0 \) and \( y'(0) = y'_0 = v_0 \)
\[
y = e^{-\alpha x}(c_1 \cos \omega x + c_2 \sin \omega x) \quad \therefore \quad 0 = c_1
\]
\[
y' = -ae^{-\alpha x}(c_1 \cos \omega x + c_2 \sin \omega x) + e^{-\alpha x}(c_2 \omega \cos \omega x - c_1 \omega \sin \omega x)
\]

\[
v_0 = c_2 \omega \quad \therefore \quad c_2 = \frac{v_0}{\omega}
\]

\[
y = e^{-\alpha x}(\frac{v_0}{\omega} \sin \omega x)
\]

Three cases of damped systems are shown below:

Curve C represents the situation when the system is overdamped as in case 1 where it takes a long time for the system to reach equilibrium.

Curve B represents critical damping as in case 2, in this case equilibrium is reached in the shortest time.

Curve A represents an underdamped situation as in case 3 in which the system makes several swings before coming to rest.

Well designed damping is needed for all kinds of applications. Large buildings in earthquake zones are now retrofitted with huge dampers to reduce earthquake damage. Shock absorbers in a car are usually designed to give critical damping, but as they wear out, under damping occurs, and they bounce up and down several times when ever it hits a bump.