DIFFERENTIATION

Definition

For any function, f(x), $f'(x) = \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

The function f(x) is said to be **differentiable** if the limit exists.

Example: Find f'(x) when $f(x) = 5x^2 - 3x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{5(x+h)^2-3(x+h)-(5x^2-3x)}{h} = \lim_{h\to 0}\frac{5(x^2+2xh+h^2)-3(x+h)-(5x^2-3x)}{h}$$

$$= \lim_{h\to 0}\frac{5x^2+10xh+5h^2-3x-3h-5x^2+3x}{h} = \lim_{h\to 0}\frac{10xh+5h^2-3h}{h} = \lim_{h\to 0}\frac{h(10x+5h-3)}{h} = \lim_{h\to 0}(10x+5h-3)$$

$$= \boxed{10x-3}$$

Note: The derivative of a constant is zero. (To check this statement, use the definition!)

Differentiation Rules

Power Rule

For $f(x) = x^n$, where n is any rational number, $f'(x) = \frac{d}{dx}\left(x^n\right) = n\cdot x^{n-1}$

Example: For $f(x) = x^4$, $f'(x) = 4\cdot x^{4-1} = \boxed{4x^3}$ or for $g(x) = \sqrt[3]{x}$, $g'(x) = \frac{d}{dx}\sqrt[3]{x} = \frac{d}{dx}x^{\frac{1}{3}} = \frac{1}{3}x^{\frac{1}{3}-1} = \boxed{\frac{1}{3}x^{-\frac{2}{3}}}$

Constant Multiple Rule

For any constant c, and differentiable function f(x), $\frac{d}{dx}\left(c\cdot f(x)\right) = c\cdot\frac{d}{dx}f(x) = c\cdot f'(x)$

Example: For $f(x) = x^3$, and c = 5, $\frac{d}{dx}\left(c\cdot f(x)\right) = \frac{d}{dx}\left(5x^3\right) = 5\cdot\frac{d}{dx}\left(x^3\right) = 5\cdot 3x^2 = \boxed{15x^2}$

Sum Rule

For any differentiable functions f(x) and g(x), $\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x)+g'(x)$

Example: For $f(x) = 2x^3$ and $g(x) = 4x^2 - 5$, $\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}2x^3 + \frac{d}{dx}\left(4x^2-5\right) = \boxed{6x^2+8x}$

Difference Rule

For any differentiable functions f(x) and g(x), $\frac{d}{dx}\left[f(x)-g(x)\right] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x) = f'(x)-g'(x)$

Example: For $f(x) = \sqrt{x}$ and $g(x) = 10x^3$, $\frac{d}{dx}\left[f(x)-g(x)\right] = \frac{d}{dx}\sqrt{x} - \frac{d}{dx}\left(10x^3\right) = \boxed{\frac{1}{2}x^{\frac{-1}{2}}-30x^2}$

Product Rule

For any differentiable functions f(x) and g(x), $\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}f(x) = f(x)\cdot g'(x)+g(x)\cdot f'(x)$

Example: For f(x) = 6x + 5 and $g(x) = x^3 - 2$, $\frac{d}{dx}\left[f(x)\cdot g(x)\right] = (6x+5)\cdot\frac{d}{dx}\left(x^3-2\right) + \left[\frac{d}{dx}(6x+5)\right]\cdot(x^3-2)$

$$= (6x+5)(3x^2)+(6)(x^3-2) = \boxed{24x^3+15x^2-12}$$

Quotient Rule

For any differentiable functions f(x) and g(x) $\quad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot\frac{d}{dx}f(x) - f(x)\cdot\frac{d}{dx}g(x)}{\left(g(x)\right)^2} = \frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{\left(g(x)\right)^2}$

Example: For $f(x) = x^2 + 2$ and g(x) = 2x – 7 $\quad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d}{dx}\left[\frac{x^2+2}{2x-7}\right] = \frac{(2x-7)\frac{d}{dx}(x^2+2)-(x^2+2)\frac{d}{dx}(2x-7)}{(2x-7)^2}$

$$= \frac{(2x-7)(2x)-(x^2+2)(2)}{(2x-7)^2} = \frac{4x^2-14x-2x^2-4}{(2x-7)^2} = \boxed{\frac{2x^2-14x-4}{(2x-7)^2}}$$

(over for more)

Differentiation Rules – (continued)

Derivatives of the Trigonometric Functions

$$\frac{d}{dx}(\sin x)=\cos x \qquad \frac{d}{dx}(\tan x)=\sec^2 x \qquad \frac{d}{dx}(\sec x)=\sec x\tan x$$

$$\frac{d}{dx}(\cos x)=-\sin x \qquad \frac{d}{dx}(\cot x)=-\csc^2 x \qquad \frac{d}{dx}(\csc x)=-\csc x\cot x$$

The Chain Rule

If y = f(u) is a differentiable function of u, and u = g(x) is a differentiable function of x, then $y=f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \quad \text{or equivalently,} \quad \frac{d}{dx}\left[f(g(x))\right]=f'(g(x))g'(x)$$

Restating this theorem using English: "To take the derivative of a composite function, take the derivative of the outside function and multiply this result by the derivative of the inside function."

The Chain Rule is an <u>enormously importantly rule</u> for taking derivatives since many functions themselves are composite functions. For example, something like f(x) = $(2x-5)^2$ is considered to be composite with f(u) = u^2 and u = g(x) = 2x – 5, as is something similar like $f(x)=\sqrt{2x-5}$.

Yet another example: $f(x)=\left(\frac{2x-3}{3x-2}\right)^4$ is a composite function with f(u) = u^4 and u = g(x) = $\frac{2x-3}{3x-2}$.

Example 1 Find the derivative of $f(x)=3\left(4-x^2\right)^5$

$$f'(x)=3\cdot\frac{d}{dx}\left(4-x^2\right)^5 \quad \text{(constant multiple rule)}$$

$$=3\left[5\left(4-x^2\right)^4\right]\cdot(-2x) \quad \text{(power rule and chain rule)}$$

$$=-30x\left(4-x^2\right)^4 \quad \text{(simplified)}$$

Example 2 Find the derivative of h(x) = $\sin^3(4x)$

$$h'(x)=3\cdot\sin^2 4x\cdot\frac{d}{dx}(\sin 4x)$$

$$=3\cdot\left[2\cdot\sin^1 4x\cdot\cos 4x\cdot\frac{d}{dx}(4x)\right]$$

$$=6\cdot\sin 4x\cdot\cos 4x\cdot(4)$$

$$=24\sin 4x\cos 4x$$

Note on Example 2 that the chain rule had to be used twice!

Implicit Differentiation

Until now on this handout, all of our functions have been written as equations with the dependent variable y or f(x) on the left hand side and with some expression containing the variable x on the right hand side. Equations such as these are said to be written in *explicit* form. Some equations containing x and y are not written this way. Consider the equation $x^2-2y^3+4y=2$. In this equation, we say that y is defined *implicitly* as a function of x. Note that it is very difficult to put this equation into explicit form, that is, to solve it for y in terms of x. Yet, we still may be interested in finding dy/dx. To find dy/dx we take the derivative of each term (you may say, take the derivative of both sides), keeping in mind that the chain rule applies! This means that since y is some function of x, whenever we take the derivative of an expression containing y, we have to then multiply it by dy/dx. After that step, we solve the resulting equation for dy/dx.

Example 1 Find dy/dx for $x^3+y^3=8$

$$\frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(y^3\right)=\frac{d}{dx}(8)$$

$$3x^2+3y^2\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-3x^2}{3y^2}=\frac{-x^2}{y^2}$$

Example 2 Find dy/dx for $x^3y^3-y=x$

(This time we will use y′ instead of dy/dx)

$3x^2y^3+x^3(3y^2)y'-1y'=1$ (Had to use the product rule)

$$3x^3y^2y'-1y'=1-3x^2y^3 \longrightarrow y'=\frac{1-3x^2y^3}{3x^3y^2-1}$$